MATH 1700: SECTION 11.1: THE PYTHAGOREAN IDENTITIES

Recall the following identities which are a consequence of the definition of the Circular Functions:

RECIPROCAL AND QUOTIENT IDENTITIES: The following relationships hold for all angles θ provided each side of each equation is defined.[1]

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$
- $\cos(\theta) = \dfrac{1}{\sec(\theta)}$
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$
- $\sin(\theta) = \dfrac{1}{\csc(\theta)}$
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$
- $\cot(\theta) = \dfrac{1}{\tan(\theta)}$
- $\tan(\theta) = \dfrac{1}{\cot(\theta)}$

Recall that the Unit Circle is described algebraically by the equation $x^2 + y^2 = 1$. Hence, for all angles, θ, $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. Conventionally, $(\cos(\theta))^2$ is written as $\cos^2(\theta)$ and $(\sin(\theta))^2 = \sin^2(\theta)$ so this identity is usually written $\cos^2(\theta) + \sin^2(\theta) = 1$. This is (one of) the so-called 'Pythagorean Identities' since it is the Pythagorean Theorem which ultimately allows us to find distance in the plane, and hence, describe circles algebraically. We list all of the variants of this identity below.

THE PYTHAGOREAN IDENTITIES:

1. $\cos^2(\theta) + \sin^2(\theta) = 1$.

   Common Alternate Forms:

   - $1 - \sin^2(\theta) = \cos^2(\theta)$
   - $1 - \cos^2(\theta) = \sin^2(\theta)$

2. $1 + \tan^2(\theta) = \sec^2(\theta)$, provided $\cos(\theta) \neq 0$.

   Common Alternate Forms:

   - $\sec^2(\theta) - \tan^2(\theta) = 1$
   - $\sec^2(\theta) - 1 = \tan^2(\theta)$

3. $1 + \cot^2(\theta) = \csc^2(\theta)$, provided $\sin(\theta) \neq 0$.

   Common Alternate Forms:

   - $\csc^2(\theta) - \cot^2(\theta) = 1$
   - $\csc^2(\theta) - 1 = \cot^2(\theta)$

[1] As an example, $\tan(0) = 0$, but $\tan(0) \neq \frac{1}{\cot(0)}$ since $\cot(0)$ is undefined.

EXAMPLE 1: Use a Pythagorean Identity and the given information to find the indicated value.

1. If θ is a Quadrant II angle with $\sin(\theta) = \frac{3}{5}$, find $\cos(\theta)$.

2. If $\pi < t < \frac{3\pi}{2}$ with $\cos(t) = -\frac{\sqrt{5}}{5}$, find $\sin(t)$.

3. If $\sin(\theta) = 1$, find $\cos(\theta)$.

4. If θ is a Quadrant IV angle with $\sec(\theta) = 3$, find $\tan(\theta)$.

5. Find $\csc(t)$ if $\pi < t < \frac{3\pi}{2}$ and $\cot(t) = 2$.

6. If θ is a Quadrant II angle with $\cos(\theta) = -\frac{3}{5}$, find the exact values of the remaining circular functions.

One of the big themes in mathematics is 'equivalence' and identities allow us to rewrite expressions involving the circular functions into forms that, depending on context, are easier to work with. (We'll see some of this when we go to solve equations and you'll see more of this if and when you enroll in a Calculus course.)

When you are asked to 'verify' an identity, the common practice is to start with one side of the identity and work to transform it into the other side of the identity using the basic identities presented in this section.

EXAMPLE 2: Verify the following identities. Assume that all quantities are defined.

1. $\tan(\theta) = \sin(\theta)\sec(\theta)$

2. $(\tan(t) - \sec(t))(\tan(t) + \sec(t)) = -1$

3. $\sin^2(x)\cos^3(x) = \sin^2(x)\left(1 - \sin^2(x)\right)\cos(x)$

4. $\dfrac{\sec(t)}{1 - \tan(t)} = \dfrac{1}{\cos(t) - \sin(t)}$

5. $6\sec(x)\tan(x) = \dfrac{3}{1 - \sin(x)} - \dfrac{3}{1 + \sin(x)}$

6. $\dfrac{\sin(\theta)}{1 - \cos(\theta)} = \dfrac{1 + \cos(\theta)}{\sin(\theta)}$

PYTHAGOREAN CONJUGATES:

- $1-\cos(\theta)$ and $1+\cos(\theta)$: $(1-\cos(\theta))(1+\cos(\theta)) = 1-\cos^2(\theta) = \sin^2(\theta)$
- $1-\sin(\theta)$ and $1+\sin(\theta)$: $(1-\sin(\theta))(1+\sin(\theta)) = 1-\sin^2(\theta) = \cos^2(\theta)$
- $\sec(\theta)-1$ and $\sec(\theta)+1$: $(\sec(\theta)-1)(\sec(\theta)+1) = \sec^2(\theta)-1 = \tan^2(\theta)$
- $\sec(\theta)-\tan(\theta)$ and $\sec(\theta)+\tan(\theta)$: $(\sec(\theta)-\tan(\theta))(\sec(\theta)+\tan(\theta)) = \sec^2(\theta)-\tan^2(\theta) = 1$
- $\csc(\theta)-1$ and $\csc(\theta)+1$: $(\csc(\theta)-1)(\csc(\theta)+1) = \csc^2(\theta)-1 = \cot^2(\theta)$
- $\csc(\theta)-\cot(\theta)$ and $\csc(\theta)+\cot(\theta)$: $(\csc(\theta)-\cot(\theta))(\csc(\theta)+\cot(\theta)) = \csc^2(\theta)-\cot^2(\theta) = 1$

STRATEGIES FOR VERIFYING IDENTITIES:

- Try working on the more complicated side of the identity.
- Use the Reciprocal and Quotient Identities to write functions on one side of the identity in terms of the functions on the other side of the identity.

  Simplify the resulting complex fractions.
- Add rational expressions with unlike denominators by obtaining common denominators.
- Use the Pythagorean Identities to 'exchange' sines and cosines, secants and tangents, cosecants and cotangents, and simplify sums or differences of squares to one term.
- Multiply numerator **and** denominator by Pythagorean Conjugates to get a difference of squares which simplifies to one term via a Pythagorean Identity.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity and see if you can find a way to bridge the two parts of your work.
- Try *something*. The more you work with identities, the better you'll get with identities.

HOMEWORK: Section 11.1: Page 912: 1 - 73 every other odd: (1, 5, 9, 13, . . . , 73)